

Queen Mary, University of London
B.Sc. Mock Examination

MAS114 Geometry I

Note: This mock examination is to show you the basic **structure** of the final examination. Many questions on this mock examination are taken directly from coursework or the mid-term test. On the final examination you should not expect questions to be exactly the same as those in coursework and/or the mid-term test (although some will be similar or based on the same concepts). Moreover, you should not expect the final examination to cover exactly the same topics as this mock paper.

2006/07

The duration of this examination is 2 hours.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Do NOT start reading the question paper until instructed to do so by the invigilator.

1. Let $A = (1, 2, 3)$ and $B = (2, -1, 4)$.

(i) [2 marks] Determine the vector represented by $\overrightarrow{AB}$.

(ii) [2 marks] Determine a vector equation for the line through the points A and B .

(iii) [4 marks] Determine a Cartesian equation for the plane through A and B and parallel to the vector $\underline{u} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

2. [6 marks] Let $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

(i) $\underline{u} \cdot \underline{v}$; (ii) $|\underline{u}|$; (iii) $\cos \theta$, where θ is the angle between $\underline{u}$ and $\underline{v}$.

3. (i) [6 marks] By reducing to echelon form, determine all solutions to the following system of linear equations in x, y, z :

$$\begin{cases} -3x & + & y & + & z & = & 2 \\ x & - & y & - & 2z & = & -1 \\ 2x & - & 2y & + & 4z & = & -2 \end{cases}.$$

(ii) [2 marks] What exactly does your answer to (i) tell you about the intersection of the three planes defined by the equations above?

4. [6 marks] Let $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 4 & 0 \\ 2 & 1 \end{pmatrix}$. For each of the following expressions, either evaluate the expression or explain why the expression has no meaning:

(i) $A + B$; (ii) AB ; (iii) BA .

Next question overleaf

5. (i) [4 marks] Calculate $\det(A)$, where $A = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 3 \\ 1 & -2 & 5 \end{pmatrix}$.

(ii) [4 marks] Find the volume of the parallelepiped with sides corresponding to $\underline{u}, \underline{v}, \underline{w}$, where $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$.

Are $\underline{u}, \underline{v}, \underline{w}$ coplanar? If $\underline{u}, \underline{v}, \underline{w}$ are not coplanar then are they a right-handed triple or a left-handed triple? Justify your answers.

6. [6 marks] Find the distance between the lines defined by $\underline{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ and $\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 6 \\ -8 \end{pmatrix}$.

7. (i) [4 marks] Define (not using co-ordinates) the *vector product* $\underline{u} \times \underline{v}$ of two vectors $\underline{u}$ and $\underline{v}$.

(ii) [6 marks] Suppose we fix the origin O , take points U, V, W so that $OUWV$ is a parallelogram, let $\underline{u}$ be the position vector of U and let $\underline{v}$ be the position vector of V . Apply your definition in (i) to prove that $|\underline{u} \times \underline{v}|$ is equal to the area of $OUWV$.

(iii) [4 marks] Assuming that for all vectors $\underline{u}, \underline{v}, \underline{w}$, we have $(\underline{u} + \underline{v}) \times \underline{w} = (\underline{u} \times \underline{w}) + (\underline{v} \times \underline{w})$, prove that for all vectors $\underline{u}, \underline{v}, \underline{w}$, we have

$$\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w}).$$

8. (i) [2 marks] State precisely what it means for an $n \times n$ matrix A to be *invertible*.

(ii) [6 marks] Prove that if A and B are invertible $n \times n$ matrices then $ABA^{-1}B^{-1}$ is invertible too.

Question continues overleaf

(iii) [4 marks] Exactly which of the following statements are true? [Your answer must be completely correct to obtain marks.]

(a) Whenever A is an invertible $n \times n$ matrix and α is a non-zero scalar, then αA is invertible.

(b) Whenever A and B are $n \times n$ matrices, such that $A + B$ is invertible, then both A and B must be invertible.

(c) Whenever A and B are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

(d) Whenever A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

9. (i) [2 marks] Define what it means for a function $t : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be a *linear transformation*.

(ii) [2 marks] Write down the 2×2 matrix S_θ representing a reflection (in the (x, y) -plane) in the line through the origin at counterclockwise angle θ from the x -axis.

(iii) [2 marks] Prove that $(S_\theta)^{-1} = S_\theta$.

(iv) [4 marks] Prove that $(S_{\theta_1}S_{\theta_2}S_{\theta_3})^{-1} = S_{\theta_3}S_{\theta_2}S_{\theta_1}$.

10. (i) [4 marks] Define what is meant by an *eigenvector* v of a linear transformation t of $\mathbb{R}^n$, and what is meant by the *eigenvalue* of t corresponding to v .

(ii) [8 marks] For each of the following matrices A , find all the eigenvalues of A , and for each such eigenvalue λ , find the set of all eigenvectors of A with corresponding eigenvalue λ :

(a) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$; (b) $\begin{pmatrix} 2 & 9 \\ 4 & -3 \end{pmatrix}$; (c) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

(iii) [6 marks] Prove that if a 2×2 matrix A has $\det(A) = 0$, then A must have an eigenvalue equal to 0.

(iv) [4 marks] Let A be an $n \times n$ matrix such that $A^2 = 0_{nn}$. Prove that if λ is an eigenvalue of A then we must have $\lambda = 0$.

End of question paper